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LETTER TO THE EDITOR

Irreducible tensor operators for the quantum algebra $su(2)_q$

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Abstract. Irreducible tensor operators of $su(2)_q$ are defined based on the coproduct is constructed by using the Jordan–Schwinger mapping.

Recently, the representations of the quantum algebra $su(2)_q$ are thoroughly discussed based on the q-harmonic oscillator realizations [1–8] for the case of q being not a root of unity. In this case the properties of quantum algebras are quite similar to those of classical Lie algebras in connection with both the representation theory and the possible applications. Besides the application in solving the Yang–Baxter equations [9]. It has been shown that rotational spectra of nuclei and molecules can be described very accurately in terms of a Hamiltonian which is proportional to the second-order Casimir operator of the quantum algebra $su(2)_q$ [10, 11]. In view of the similar properties of quantum algebras to those of classical algebras, it may also be possible to define tensor operators, reduced matrix elements, etc, for quantum algebras.

The quantum algebra $su(2)_q$ is generated by J_+, J_- and J_0 under the relations

$$[J_0, J_\pm] = \pm J_\pm \tag{1}$$

$$[J_+, J_-] = [2J_0] \tag{2}$$

where for given x

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{3}$$

which reduce to the usual relations among generators of su(2) in the limit $q \to 1$.

An algebra homomorphism (coproduct) $\Delta: su(2)_q \to su(2)_q \oplus su(2)_q$ reads

$$\Delta(J_0) = J_0 \otimes 1 \oplus 1 \otimes J_0 \tag{4a}$$

$$\Delta(J_\pm) = J_\pm \otimes q^{+J_0} \oplus q^{-J_0} \otimes J_\pm . \tag{4b}$$

Let us consider an associative boson algebra $\mathscr{B}(q)$ with units generated by three elements b, b^+ and N which satisfy the relation

$$[N, b^+] = b^+ \qquad [N, b] = -b \tag{5}$$

$$bb^+ - q^{\pm 1} b^+ b = q^{\mp N}. \tag{6}$$

To introduce star operation in $\mathscr{B}(q)$ we suppose that q is real,

$$(b)^\dagger = b^+ \qquad (b^+)^\dagger = b \qquad N^\dagger = N. \tag{7}$$

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Then the relations (5) and (6) are invariant under the star anti-involution. Other useful relations are

$$q^{\pm N}b^{+}q^{\mp N}=b^{+}q^{\pm 1} \qquad q^{\pm N}bq^{\mp N}=bq^{\mp 1} \tag{8}$$

which can be derived from (5).

The Jordan–Schwinger realizations of $su(2)_q$ has been given by many authors [4-6]. Using two kinds of q-deformed boson operators (b_i^+, b_i, N_i, $i=1,2$), one can define

$$J_{+}=b_1^{+}b_2 \qquad J_{-}=b_2^{+}b_1 \tag{9}$$

and find

$$[J_+, J_-]=[2J_0] \tag{10}$$

where

$$J_0=N_1-N_2. \tag{11}$$

The related realizations of the basis vectors $|jm\rangle_q$ of $su(2)_q$ can be defined as

$$|jm\rangle_q=\frac{b_1^{+j+m}b_2^{+j-m}}{\{[j+m]![j-m]!\}^{1/2}}|0\rangle. \tag{12}$$

We assume that $T_m^k(q)$ is an irreducible tensor operator of rank k for $su(2)_q$; $|jm'\rangle_q$ is any basis vector of $su(2)_q$. Then, from the coproduct definition we have

$$J_{\pm}T_m^k(q)|jm'\rangle_q=(J_{\pm}T_m^k(q))q^{J_0}|jm'\rangle_q+T_m^k(q)q^{-m}J_{\pm}|jm'\rangle_q \tag{13}$$

and

$$J_0T_m^k(q)|jm'\rangle_q=(J_0T_m^k(q))|jm'\rangle_q+T_m^k(q)J_0|jm'\rangle_q \tag{14}$$

which gives

$$(J_{\pm}T_m^k(q))=(J_{\pm}T_m^k(q)-T_m^k(q)q^{-m}J_{\pm})q^{-J_0} \tag{15a}$$

$$(J_0T_m^k(q))=J_0T_m^k(q)-T_m^k(q)J_0. \tag{15b}$$

Because $T_m^k(q)$ is an irreducible tensor operator, any basis vectors $|jm\rangle_q$ can also be written as

$$|jm\rangle_q=T_m^j(q)|\tilde{0}\rangle \tag{16}$$

where $|\tilde{0}\rangle$ is the basis vector of identity representation

$$J_i|\tilde{0}\rangle=0 \qquad \text{for } i=+,-,0. \tag{17}$$

We have

$$\begin{aligned}J_{\pm}|jm\rangle_q&=(J_{\pm}T_m^j(q))|\tilde{0}\rangle=\{[j\mp m][j\pm m+1]\}^{1/2}|jm\pm 1\rangle_q\\&=\{[j\mp m][j\pm m+1]\}^{1/2}T_{m\pm 1}^j(q)|\tilde{0}\rangle\end{aligned} \tag{18a}$$

$$J_0|jm\rangle_q=(J_0T_m^j(q))|\tilde{0}\rangle=m|jm\rangle_q=mT_m^j(q)|\tilde{0}\rangle. \tag{18b}$$

Combining equations (15) and (18) we obtain

$$(J_{\pm}T_m^k(q)-T_m^k(q)q^{-m}J_{\pm})q^{-J_0}=\{[k\mp m][k\pm m+1]\}^{1/2}T_{m\pm 1}^k(q) \tag{19a}$$

$$[J_0, T_m^k(q)]=mT_m^k(q). \tag{19b}$$

This can be taken as the definition of the irreducible tensor operators of $su(2)_q$.

Using the definition (19) and Jordan–Schwinger realizations of $su(2)_q$, we can directly prove that both

$$T^k_m(q)=\frac{1}{\{[k+m]![k-m]!\}^{1/2}}\,b_1^{+k+m}b_2^{+k-m}q^{N_1(k-m)/2-N_2(k+m)/2} \tag{20}$$

and

$$V^k_m(q)=\frac{(-1)^{k-m}}{\{[k+m]![k-m]!\}^{1/2}}\,b_1^{k-m}b_2^{k+m}q^{N_1(k+m)/2-N_2(k-m)/2-(k-m)} \tag{21}$$

are irreducible tensor operators of rank k for $su(2)_q$.

From equations (20) and (21) we obtain

$$(V^k_m(q))^{\dagger}=(-)^{k-m}q^{-(k-m)}T^k_{-m}(q) \tag{22}$$

which also contract to the conjugation relation of irreducible tensor operators of su(2) when $q\to 1$.

Using the Wigner–Eckart theorem, we have

$${}_q\langle J'm'|T^k_m(q)|jm''\rangle_q=\langle jm''km|j'm'\rangle_q\langle j'\|T^k(q)\|j\rangle_q \tag{23}$$

where $\langle j'\|T^k(q)\|j\rangle_q$ is the $su(2)_q$ reduced matrix element, and $\langle jm''km|j'm'\rangle_q$ is the $su(2)_q$ CG coefficient.

Let $D^{j_1j_2}_q$ denote q-representation obtained from the coupling $D^{j_1}_q\otimes D^{j_2}_q$, then the transpose P_{12} satisfies

$$P_{12}D^{j_1j_2}_q=D^{j_2j_1}_qP_{12} \tag{24}$$

which gives the symmetry properties of $su(2)_q$ CG coefficient

$$\begin{aligned}
&\langle j_1m_1j_2m_2|jm\rangle_q\\
&\quad=(-1)^{j_1+j_2-j}\langle j_1-m_1j_2-m_2|j-m\rangle_{q^{-1}}\\
&\quad=(-1)^{j_1+j_2-j}\langle j_2m_2j_1m_1|jm\rangle_{q^{-1}}\\
&\quad=(-1)^{j_1-m_1}q^{-m_1}\left\{\frac{[2j+1]}{[2j_2+1]}\right\}^{1/2}\langle j_1m_1j-m|j_2-m_2\rangle_{q^{-1}}\\
&\quad=(-1)^{j_2+m_2}q^{m_2}\left\{\frac{[2j+1]}{[2j_1+1]}\right\}^{1/2}\langle f-mj_2m_2|j_1-m_1\rangle_{q^{-1}}.
\end{aligned} \tag{25}$$

Taking the matrix element of (22) and using the Wigner–Eckart theorem and symmetry properties given by (25), we obtain

$$\langle j\|V^k(q)\|j'\rangle^*_q=(-1)^{j'+k-j}q^{-k}\left\{\frac{[2j'+1]}{[2j+1]}\right\}^{1/2}\langle j'\|T^k(q)\|j\rangle_q. \tag{26}$$

Finally, we give some reduced matrix elements of $T^k_m(q)$ and $V^k_m(q)$

$$\begin{aligned}
&\langle j+\tfrac{1}{2}\|T^{1/2}(q)\|j\rangle_q=([2j+1])^{1/2}\\
&\langle j-\tfrac{1}{2}\|V^{1/2}(q)\|j\rangle_q=-(q^{-1}[2j+1])^{1/2}
\end{aligned} \tag{27}$$

for which the corresponding CG coefficients are

$$\begin{aligned}
&\langle jm\ \tfrac{1}{2}\pm\tfrac{1}{2}|j+\tfrac{1}{2}\ m\pm\tfrac{1}{2}\rangle_q=q^{\mp(j\mp m)/2}\left(\frac{[j\pm m+1]}{[2j+1]}\right)^{1/2}\\
&\langle jm\ \tfrac{1}{2}\pm\tfrac{1}{2}|j-\tfrac{1}{2}\ m\pm\tfrac{1}{2}\rangle_q=\pm q^{\pm(j\pm m+1)/2}\left(\frac{[j\mp m]}{[2j+1]}\right)^{1/2}
\end{aligned} \tag{28}$$

and

$$\langle j+k\| T^k(q)\| j\rangle_q = \left(\frac{[2k+2j]!}{[2k]![2j]!}\right)^{1/2}$$
$$\langle j-k\| V^k(q)\| j\rangle_q = q^{-k}(-1)^{2k}\left(\frac{[2j+1]!}{[2k]![2j-2k+1]!}\right)^{1/2} \qquad (29)$$

for which the corresponding CG coefficients are

$$\langle jm\ \ kp|j+k\ \ m+q\rangle_q = q^{(k-p)(j+m)/2-(k+p)(j-m)/2} \times\left(\frac{[2k]![2j]![k+p+j+m]![j+k-p-m]!}{[k+p]![k-p]![j+m]![j-m]![2k+2j]!}\right)^{1/2}$$

$$\langle jm\ \ kp|j-k\ \ m+p\rangle_q = q^{(k+p)(j+m)/2-(k-p)(j-m)/2+p}(-1)^{k+p} \times\left(\frac{[2k]![2j-2k+1]![j+m]![j-m]!}{[k+p]![k-p]![j-k-m-p]![j-k+m+p]![2j+1]!}\right)^{1/2}. \qquad (30)$$

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